1-10的奇數題，11-20的偶數題.

9.1

Pf: $f(0)=0$ & $f(x)$ is continuous at 0.

9.3

Pf: $f(x)\equiv$ constant

Suppose that f is a periodic function s.t.

$$\begin{cases} f(x+T)=f(x),\ \forall x\in\mathbb{R} \text{ for some } T>0 \\ \lim_{x\to+\infty} f(x)=C \text{ exists}. \end{cases}$$

Then

$$\begin{aligned} & f(x) \\ = \ & f(x+nT) \quad \forall n\in\mathbb{N} \\ = \ & \lim_{n\to+\infty} f(x+nT) \\ = \ & \lim_{x\to+\infty} f(x) = C \quad \forall x\in\mathbb{R} \end{aligned}$$

□.

9.5.

9.7

Pf: (1). ① $x\in\mathbb{N}$

$f(0)=f(0)+f(0) \Rightarrow f(0)=0$

$\forall x\geq 1,\ f(x)=f(\sum_{k=1}^{x} 1)=\sum_{k=1}^{x} f(1)=x\cdot\alpha$

② $x\in\mathbb{Z}$.

If $x\leq 0$, then $f(x)+f(-x)=f(0) \Rightarrow f(x)=f(0)-f(-x) \overset{-x\in\mathbb{N}}{=} f(0)-\alpha\cdot(-x)=\alpha x$

③ $x=\frac{p}{q}\in\mathbb{Q}$

$f(p)=f(q\cdot\frac{p}{q})=q\cdot f(\frac{p}{q}) \Rightarrow f(\frac{p}{q})=\frac{1}{q}f(p)=\frac{1}{q}\cdot p\cdot\alpha=\frac{p}{q}\alpha$

(2) Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\forall x \in \mathbb{R}$, $\exists \{x_n\} \subset \mathbb{Q}$ s.t. $x_n \to x$, then

$$f(x) = \lim_{x_n \to x} \alpha \cdot x_n = \alpha \cdot x.$$

9.9 Pf: Define $f(x) = |x-n|$, $x \in [-\frac{1}{2}+n, \frac{1}{2}+n[$ $(n \in \mathbb{Z})$, you only to check that f is continuous at $x = \frac{1}{2}+n$, $\forall n \in \mathbb{Z}$.

9.12.

Pf: $\frac{x}{x+1} \cdot \sin x$

9.14

Pf: Since $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = +\infty$, $\exists M > 0$ s.t.

$$f(x) > f(0), \quad \forall |x| > M$$

$$\Rightarrow f(x) > \inf_{x \in [-M, M]} f(x), \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \inf_{x \in \mathbb{R}} f(x) = \inf_{x \in [-M, M]} f(x)$$

which can be achived at some point $x_0 \in [-M, M]$.

9.16.

Pf: Since f, g is continuous, $\forall x_1, x_2 \in \mathbb{R}$, $\exists t_1, t_2 \in [0,1]$ s.t.

$$\varphi(x_1) = f(t_1) + x_1 g(t_1)$$

$$\varphi(x_2) = f(t_2) + x_2 g(t_2).$$

$$\begin{aligned}
\varphi(x_1) - \varphi(x_2) &= f(t_1) + x_1 g(t_1) - f(t_2) - x_2 g(t_2) \\
&= f(t_1) + x_2 g(t_1) - f(t_2) - x_2 g(t_2) + x_1 g(t_1) - x_2 g(t_1) \\
&= f(t_1) + x_2 g(t_1) - \sup_{t \in [0,1]} \left(f(t) + x_2 g(t) \right) + (x_1 - x_2) g(t_1) \\
&\leq |x_1 - x_2| \cdot \sup_{[0,1]} |g(t)|
\end{aligned}$$

$$\varphi(x_1)-\varphi(x_2) = f(t_1)+x_1 g(t_1) - f(t_2) - x_2 g(t_2)$$

$$= \sup_{[0,1]} f(t)+x_1 g(t) - \left(f(t_2)+x_1 g(t_2)\right) + (x_1-x_2)\, g(t_2)$$

$$\geq -\sup_{[0,1]} |g(t)| \cdot |x_1-x_2|$$

$$\Rightarrow \quad |\varphi(x_1)-\varphi(x_2)| \leq \sup_{[0,1]} g \cdot |x_1-x_2| \,, \quad \forall x_1, x_2 \in \mathbb{R}$$

□

9.18

Pf: Define $f: \begin{cases} 0 & \mathbb{Q} \\ 1 & [0,1]\setminus \mathbb{Q} \end{cases}$

9.20.

Pf: f is uniformly continuous $\Rightarrow$ For $\varepsilon=1$, $\exists\, \delta>0$ s.t. $\forall\, |x_1-x_2|<\delta$, we have $|f(x_1)-f(x_2)|<1$.

$\forall x \geq 0$, $\exists\, k \in \mathbb{N}$ s.t. $x \in [k\delta, (k+1)\delta]$, then

$$|f(x)-f(0)|$$

$$\leq |f(x)-f(k\delta)| + \left|\sum_{n=0}^{k-1} f((k+1)\delta) - f(k\delta)\right|$$

$$\leq 1+k\cdot 1 = k+1 = \frac{k+1}{k\delta}\cdot k\delta \leq \frac{k+1}{k\delta}\cdot x \leq \frac{x}{\delta}$$

In a similar way, $\forall x<0$, $|f(x)-f(0)| \leq \frac{x}{\delta}$.

$$\Rightarrow \quad |f(x)| \leq |f(0)| + \frac{|x|}{\delta}$$

□.

Remark:

f is uniformly continuous $\not\Rightarrow$ f is lip. eg: $\sqrt{|x|}$